Homotopy theory – 同伦理论

Melchior Grützmann / 古梅西
www.grutzmann.de/geometry

2012 年 4 月 25 日

Outline

Literature

- J. MAY: *A Concise Course in Algebraic Topology* (Univ. Chicago Press, 1999), ISBN 978-0226511832.
- J. MUNKRES: *Topology*, 2nd edn., 537 pp. (Prentice Hall, 2000), ISBN 0131816292.
- WRITTEN BY THE WEB: *Wikipedia the free encyclopedia*. http://www.wikipedia.org/.

Homotopic maps

Definition

Two continuous maps $f, g\colon X \rightrightarrows Y$ are called homotopic $f \simeq g$ iff there is a continuous map $F\colon [0,1] \times X \to Y$ such that $F(0,\bullet) = f$ and $F(1,\bullet) = g$.

Example

The map $f\colon [0,1] \to [0,1] : x \mapsto x$ and $g\colon [0,1] \to [0,1] : x \mapsto 0$ are homotopic via $F\colon [0,1] \times [0,1] \to [0,1] : (t,x) \mapsto tx$ which is continuous and $F(0,x) = g(x)$, $F(1,x) = f(x)$.

Remark

Homotopy is an equivalence relation (等价关系; reflexive – 自反, symmetric – 对称 (chèn), transitive – 传递 (chuándì)).

Homotopic spaces

Definition

Two topological spaces $(X, \mathcal{O})$ *and* $(Y, \mathcal{P})$ *are said to be homotopic* $X \simeq Y$ *iff there are two continuous maps* $f\colon X \to Y$ *and* $g\colon Y \to X$ *such that* $f \circ g \simeq \mathrm{id}_Y$ *and* $g \circ f \simeq \mathrm{id}_X$*.*

Example

1. The spaces $[0,1]$ and $\{\mathrm{pt}\}$ are homotopic via the maps $f\colon [0,1] \to \{\mathrm{pt}\} : x \mapsto \mathrm{pt}$ and $g\colon \{\mathrm{pt}\} \to [0,1] : \mathrm{pt} \mapsto 0$, because $f \circ g\colon \{\mathrm{pt}\} \to \{\mathrm{pt}\} : \mathrm{pt} \mapsto \mathrm{pt}$ is the identity $\mathrm{id}_{\mathrm{pt}}$ and $g \circ f\colon [0,1] \to [0,1] : x \mapsto 0$ is homotopic to $\mathrm{id}_{[0,1]}$ as we have seen before.
2. Classify the capital letters of the pinyin alphabet up to homotopy: A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z
   $\{A, D, O, P, Q, R\}$, $\{B\}$, $\{C, E, F, G, H, I, J, K, L, M, N, S, \dots\}$.

Homotopy is an equivalence relation.

Fundamental group (基本群) I

jī

Q: What is the difference between a sphere and a torus?
a: Try fishing with a rope! On the torus your rope might get stuck.

Definition

Given a path-connected topological space $(X, \mathcal{O})$. The fundamental group $\pi_1(X)$ based at $x \in X$ is the set of continuous maps from the circle $[0,1]/\sim$ to X such that 0 and 1 are both mapped to x, modulo homotopy with fixed endpoints. The multiplication is concatenation of paths (路径的串联), i.e. $g, h\colon [0,1] \rightrightarrows X$, then $[g] * [h] := [p]$ with

chuànlián

$$p\colon [0,1] \to X : t \mapsto \begin{cases} g(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ h(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases} \tag{1}$$

Fundamental group II

Proposition

The multiplication is indeed well defined and makes $\pi_1(X)$ into a discrete group. This group is independent of the base point as long as X is path-connected.

Proof. Let $f_0 \xrightarrow[H]{\sim} f_1$ and $g_0 \xrightarrow[H']{\sim} g_1$, then $f_0 * g_0 \xrightarrow[H*H']{\sim} f_1 * g_1$ with

$$(H * H')(t,x) = \begin{cases} H(2t,x) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ H'(2t-1,x) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Fundamental group III

We need to check associativity (关联 lián), i.e. let $f, g, h\colon [0,1] \overset{\to}{\underset{\to}{\to}} X$ be three paths and consider

$$((f * g) * h)(t) = \begin{cases} f(4t) & \text{for } 0 \leq t \leq \frac{1}{4}, \\ g(4t-2) & \text{for } \frac{1}{4} \leq t \leq \frac{1}{2}, \\ h(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

while

$$(f * (g * h))(t) = \begin{cases} f(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ g(4t-2) & \text{for } \frac{1}{2} \leq t \leq \frac{3}{4}, \\ h(4t-3) & \text{for } \frac{3}{4} \leq t \leq 1. \end{cases}$$

It is not hard to see that this is homotopic by stretching the first quarter and shrinking the last half of $[0,1]$.

Fundamental group IV

The neutral element (中性元素 sù) is the constant map, i.e. $\mathrm{pt}\colon [0,1] \to X : t \mapsto x$.

The inverse element (逆元素 Nì) of $f\colon [0,1] \to X$ is $\bar{f}\colon [0,1] \to X : t \mapsto f(1-t)$.

To see that $\pi_1(X)$ is "independent of the base point" (基点独立 Jī dú) $x \in X$, note that for another base point $y \in X$, there is a path $\gamma\colon [0,1] \to X$ with $\gamma(1) = x$ and $\gamma(0) = y$. But this translates every element $[f] \in \pi_1(X)_x$ into the element $[\gamma * f * \bar{\gamma}] \in \pi_1(X)_y$. This is a group homomorphism (群同态 tài), because $\bar{\gamma} * \gamma \simeq \mathrm{pt}$ and the inverse to conjugation with γ is just conjugation with $\bar{\gamma}$. □

Example

1. Given the circle $\mathbb{S}^1$, then the fundamental group contains a subgroup generated by winding (蜿蜒) once around the circle $\gamma\colon [0,1] \to \mathbb{S}^1 : t \mapsto \mathrm{e}^{2\pi i t}$.

   With the concept of winding number (蜿蜒数 Wānyán) it becomes clear that every closed path in $\mathbb{S}^1$ can be contracted to an integer multiple of γ, but γ or any of its non-zero multiples cannot be contracted to the constant map. Therefore $\pi_1(\mathbb{S}^1) = \langle \gamma \rangle \cong \mathbb{Z}$.

Fundamental group – Example 2

Consider the sphere (球面) $\mathbb{S}^2$. Trying with pencil and paper it seems impossible to draw a closed curve on it that cannot be contracted to a point. Given the space $\mathbb{R}^2$ which is the sphere without the north pole, we can contract the whole space to a point and therefore all loops in $\mathbb{S}^2$ that leave out one point, are homotopic to the constant map pt.

But there are also sphere filling curves! E.g. the dragon curve (龙形曲线 qūxiàn) on the sphere. The answer to that is one can approximate every continuous loop with a differentiable loop. This approximation can be realized as a homotopy. But the differentiable curves have measure dimension 1 while that of the sphere is 2. Therefore they leave out at least one point.

Fundamental group – Example 3

Consider the torus $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$. Drawing a bit with pencil and paper, we see that there are 2 seemingly non-homotopic simple loops – one along the outer rim and one orthogonally to it along the smaller circumference. Therefore $\pi_1(\mathbb{T}^2)$ seems to contain a quotient of $F_2 := \langle a, b \rangle$, the free group (自由群) in 2 generators (生成元). Visualizing the torus as a square where the opposite egdes are identified while keeping the orientation, it becomes apparent that the loops a^n or b^n cannot be contracted to the constant map unless $n = 0$. But there is also another relation between them. Choose the corners of the square as base point. Then walking around the boundary of the square corresponds to the loop $aba^{-1}b^{-1}$ which via the face of the square is contractible to the constant map pt. The relation $aba^{-1}b^{-1} = 1$ is equivalent to $ab = ba$ which tells

huàn

us that the two generators commute (交换). Therefore the subgroup generated by a and b is actually $\mathbb{Z} \oplus \mathbb{Z}$.

Using arguments analogously to the sphere example, we can show that every loop in $\mathbb{T}^2$ can be contracted to $a^m b^n$ with $m, n \in \mathbb{Z}$ and therefore $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$.

Simple connectedness (简单连接)

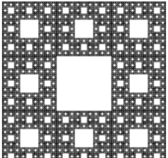
Jiǎndān jiē

Definition

Given a path connected topological space X. We say that X is simply connected iff $\pi_1(X) = 1$, the trivial group.
A topological space X is called locally (本地) simply connected iff for every point $x \in X$ and every open neighborhood $x \in U_x \subset X$ there is an open subset $x \in V_x \subset U_x$ that is path-connected and simply connected.

Example

1. The sphere $\mathbb{S}^2$ is simply connected and also locally simply connected. The latter because the open disk $\mathbb{D}$ is homotopic to a point.
2. Neither the circle $\mathbb{S}^1$ nor the torus $\mathbb{T}^2$ are simply connected, but they are both locally simply connected.



The Sierpiński carpet (谢尔宾斯基 地毯) is path connected, but not locally simply connected.

sījī tǎn

Universal cover (Fàn 泛覆 fùdié 叠空间)

Definition

Given a surjective (Mǎnshè 满射) continuous map $p\colon E \twoheadrightarrow X$. It is called a covering map iff for every point $x \in X$, there are is an open neighborhood $x \in V_x \subset X$ and for every $e \in E_x$, $e \in U_e \subset E$ such that $p\colon U_e \twoheadrightarrow V_x$ is a homeomorphism. $p\colon E \to X$ is called (path) connected iff E is (path) connected.

Example

Consider the projection (Tóuyǐng 投影) $p\colon \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z} \approx \mathbb{S}^1$. $\mathbb{R}$ is simply connected (because it is homotopic to a point) and the fiber over a point $x + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ is just $\{x + z : z \in \mathbb{Z}\} \approx \mathbb{Z}$ a discrete space.

Theorem

Given a path-connected covering map $p\colon E \to X$, then the fundamental group $\pi_1(X)$ acts transitively on the fibers $p^{-1}(x)$ for every $x \in X$ and the kernel of this action is $\ker \rho \cong \pi_1(E)$.
In particular given a path-connected locally simply connected topological space X, there is a universal cover $\tilde{X}$ which is simply connected, $X \approx \tilde{X}/\pi_1(X)$, and $\tilde{X} \twoheadrightarrow X$ factors through every cover $p\colon X' \twoheadrightarrow X$ as $\widetilde{X'} \cong \tilde{X} \twoheadrightarrow X' \twoheadrightarrow X$.

Idea of proof

Covering map implies that the fibers (Xiānwéi 纤维) $E_x := p^{-1}(x) \subset E$ for $x \in X$ are all discrete spaces. Given an element $[\gamma] \in \pi_1(X)$ based at x and an element $e \in E_x$, by assumption we have open neighborhoods $e \in U_e \subset E$ and $x \in V \subset X$ that are homeomorphic via p. Therefore we can locally lift (tíshēng 提升) $\gamma \cap V$ to $\tilde{\gamma} \cap U_e$. Covering all of γ with such neighborhoods V and choosing the e consistently along γ, we can lift all of γ to a path $\tilde{\gamma}$ in E from e to a point in E_x. This is called lift of the curve γ through e. It is unique and gives the action of $\pi_1(X)$ on E_x. Since E is path-connected, the action is transitive. This proves the first part.
The construction of the universal cover goes along the locally simply connected neighborhoods, i.e. $X_0 := \bigsqcup_{V_x \subset X} \pi_1(X) \times V_x$ glued together along the intersections $V_x \cap V_y$.

Remark

If X is in addition Hausdorff, then also $\tilde{X}$ is Hausdorff.

Examples of universal cover

1. Given the space $\mathbb{S}^1 \approx \mathbb{R}/\mathbb{Z}$, then its fundamental group is $\mathbb{Z}$ and the universal cover $p\colon \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$.
2. Given the torus $\mathbb{T}^2$, then its fundamental group is $\pi_1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$ and given the construction from the unit square it is not hard to see that $\widetilde{\mathbb{T}^2} = \mathbb{R}^2$ with the projection $p\colon \mathbb{R}^2 \twoheadrightarrow \mathbb{T}^2 : (x, y) \mapsto (x + \mathbb{Z}, y + \mathbb{Z})$.
3. Consider the join of a sphere and a circle $X := \mathbb{S}^2 \vee \mathbb{S}^1$. Since the sphere is simply connected, we only need to use the universal cover of $\mathbb{S}^1$ which is $\mathbb{R}$. Therefore the universal cover of X is a line with infititely many spheres attached. This is also homotopic to an infinite stack of touching spheres.

Groupoids (群胚)

Q: What if the space X is not path connected?

a: You should choose a base point per path-connected component.

Definition

A groupoid is a septuple (X, G, s, t, m, e, i) *where* X *and* G *are sets,* $s, t\colon G \rightrightarrows X$ *are maps called source and target,* $G_0 := X$, $G_n := \{(g_1, g_2, \ldots, g_n) \in G^n : s(g_i) = t(g_{i+1}) \ \forall i = 1, \ldots, n-1\}$ *for* $n \in \mathbb{N}^*$. $m\colon G_2 \to G$, $e\colon X \to G$ *and* $i\colon G \to G$ *subject to the rules*

$$s(e(x)) = x = t(e(x)), \tag{2}$$

$$m(e(t(g)), g) = g = m(g, e(s(g))), \tag{3}$$

$$t(m(g,h)) = t(g), \quad s(m(g,h)) = s(h), \tag{4}$$

$$m(g, m(h,k)) = m(m(g,h), k), \tag{5}$$

$$t(i(g)) = s(g), \quad s(i(g)) = t(g), \tag{6}$$

$$m(g, i(g)) = e(t(g)), \quad m(i(g), g) = e(s(g)). \tag{7}$$

Examples of groupoids

Remark

The elements $x \in X$ are called base points. The elements $g \in G$ are called arrows, they are pointing from the source $s(g)$ to the target $t(g)$. The groupoid multiplication is also written as $m(g, h) = gh$. $e(x)$ are called the units and $i(g)$ the inverse elements.

Example

0. Given a group G, then this is a groupoid over $X = \{\mathrm{pt}\}$ with the sole unit $e(\mathrm{pt}) = 1$ and the obvious source and target maps $s = t = \mathrm{pt}$. Therefore $G_n = G^n := G \times \cdots \times G$ (n components G), m the group multiplication and $i(g) = g^{-1}$ the inverse element.
1. Let $X = G$ be any set and $s = t = e = \mathrm{id}_X$. Then $G_n = \Delta_X^n$ (the n-fold diagonal – 对角) and therefore $m = \mathrm{pr}\colon \Delta_X^2 \xrightarrow{\simeq} X$ as well.

Fundamental groupoid

Definition

Given a topological space X (that is locally simply connected), then $\Pi(X)$ is the groupoid over X with total space G the continuous paths in X modulo homotopy with fixed endpoints. Source and target map are the end points of the path. Multiplication is done via concatenation of paths.

Corollary

Given a topological space that is locally simply connected, then $\Pi(X)$ is a topological groupoid with discrete stabilizers $\Pi(X)_x := s^{-1}(x) \cap t^{-1}(x)$ for every $x \in X$.

This follows analogously to the proof for the fundamental group.

Remark

A groupoid $(G \rightrightarrows X)$ is called transitive iff for every pair of points $x, y \in X$ there is an arrow $g \in G$ with $x = s(g)$ and $y = t(g)$.
In particular the fundamental groupoid $\Pi(X)$ is transitive iff X is path-connected.

Examples of fundamental groupoids

1. Given a path-connected space X, then the fundamental groupoid is $G \cong \tilde{X} \times \tilde{X}/\pi_1(X)$.
2. If X is not path-connected then $\Pi(X)$ is the disjoint union of the path-conneced-components, i.e.

$$\Pi(X) \cong \bigsqcup_{U \in [X]} \Pi(U)$$

where $[X]$ denotes the path-connected components of X.

Interludium: Free products of groups I

In order to formulate the gluing theorem for fundamental groups, we first need to expose some algebraic structure involved.

Definition

Given two groups G_k, $k = 1, 2$, then their free product is a group denoted $G_1 * G_2$ together with two injective group homomorphisms $i_k\colon G_k \to G_1 * G_2$ such that for every pair of group homomorphisms $j_k\colon G_k \to H$ there is a unique group homomorphism $j\colon G_1 * G_2 \to H$ with $j_k = j \circ i_k$.

Proposition

Given two discrete groups G_k, $k = 1, 2$, then there exists a free product which is unique up to unique isomorphism.

Interludium: Free products of groups II

Idea of proof.

Uniqueness of the free product follows from its universality property. The harder part is to argue formally for the existence of the free product.

First note that there are free groups $F(S)$ which given a set S describe the group on its symbols without any relations except for associativity. The formal construction is via the fully canceled finite words over the alphabet $\Sigma := S \cup \bar{S}$.

Assume that the groups G_k are finitely generated, say by the disjoint generators $S_k \subset G_k$, $S_1 \cap S_2 = \emptyset$ with the relations $N_k \lhd F(S_k)$, i.e. $1 \to N_k \to F(S_k) \to G_k \to 1$ is an exact sequence.

Then we can construct the free product as $G_1 * G_2 \cong F(S_1 \cup S_2)/\langle N_1 \cup N_2 \rangle$ where $\langle \ldots \rangle \lhd F(\ldots)$ is the normal subgroup generated by all elements . . . . □

Seifert-van Kampen theorem I

Q: Given a topological space $X = A \cup B$ that is the union of two subspaces. Suppose further that we are able to compute the fundamental groups of each component as well as their intersection. What can we say about the fundamental group of the total space?

Theorem

Given two connected open subspaces $U_k \subset U_1 \cup U_2$, $k = 1, 2$ with connected intersection $U_1 \cap U_2$, then the fundamental group of their union computes as

$$\pi_1(U_1 \cup U_2, x) \cong \pi_1(U_1, x) * \pi_1(U_2, x)/N(U_1, U_2) \tag{8}$$

where $x \in U_1 \cap U_2$, $N(U_1, U_2) \triangleleft \pi_1(U_1, x) * \pi_1(U_2, x)$ is the normal subgroup generated by $i_1 \circ b_1(g^{-1}) i_2 \circ b_2(g)$ for all $g \in \pi_1(U_1 \cap U_2, x)$, $i_k \colon \pi_1(U_i, x) \to \pi_1(U_1, x) * \pi_1(U_2, x)$, and $b_k \colon \pi_1(U_1 \cap U_2) \to \pi_1(U_k)$.

Proof.

Since the four spaces are path-connected topological spaces, their fundamental groups are well defined. Moreover $j_k\colon U_k \hookrightarrow U_1 \cup U_2$ implies a group homomorphism $j_{k*}\colon \pi_1(U_k, x) \to \pi_1(U_1 \cup U_2, x)$. Note that j_{k*} may be neither surjective nor injective. Therefore there exists a map $j\colon \pi_1(U_1) * \pi_1(U_2) \to \pi_1(U_1 \cup U_2)$ whose kernel contains all those elements $g_{1,1}g_{1,2}g_{2,1}\ldots g_{n,2}$ with $g_{i,k} \in \pi_1(U_k)$ that vanish in $\pi_1(U_1 \cup U_2)$. For $n = 2$ this implies $j_1(g_{1,1}) = j_2(g_{1,2}^{-1}) \in j_1(\pi_1(U_1)) \cap j_2(\pi_1(U_2))$. But this implies that there is a path $\gamma\colon [0,1] \to U_1 \cap U_2$ with $g_{1,1} = [\gamma]_1 \in \pi_1(U_1)$ and $g_{1,2}^{-1} = [\gamma]_2 \in \pi_1(U_2)$. Therefore $g := [\gamma]_{1\wedge 2} \in \pi_1(U_1 \cap U_2)$ has $b_1(g) = g_{1,1}$ and $b_2(g) = g_{1\wedge 2}^{-1}$. MG: By analogous means one can see that the higher products are just products of such elements.

To argue that j is surjective, note that every path $\gamma\colon [0,1] \to U_1 \cup U_2$ can be covered by open intervals $V_{i,k} \subset [0,1]$ such that $\gamma(V_{i,k}) \subset U_k$ and only the adjacent $V_{i,1}$ and $V_{i,2}$ or $V_{i,2}$ and $V_{i+1,1}$ overlap. Since $[0,1]$ is compact, finitely many $i = 1,\ldots,n$ are enough. By shrinking the $V_{i,k}$ if necessary we can ensure that they still cover $[0,1]$ and their closures are still mapped to U_k each. Since the $\gamma(V_{i,1} \cap V_{i,2}) \subset U_1 \cap U_2$, we can pick $x_i \in V_{i,1} \cap V_{i,2}$ and change γ by a homotopy such that $\gamma(x_i) = x$. By analogous means also $y_i \in V_{i,2} \cap V_{i+1,1}$ with $\gamma(y_i) = x$. But this implies $[\gamma]_{1\vee 2} = j_1(g_{1,1})j_2(g_{1,2})\ldots j_2(g_{n,2})$ for $g_{i,1} := [\gamma|[y_{i-1}, x_i]]_1 \in \pi_1(U_1)$ and analogous for $g_{i,2} \in \pi_1(U_2)$. □

Push out

The above construction fits into the following commutative square:

$$\begin{array}{ccc} \pi_1(U_1 \cup U_2) & \dashrightarrow & \pi_1(U_2) \\ \downarrow & & \downarrow \\ \pi_1(U_1) & \longrightarrow & \pi_1(U_1 \cap U_2) \end{array}$$

It is therefore called push-out or free fibered product of groups. In the category of discrete groups these always exist.

Remark

It is also possible to carry over this construction to the category of groupoids. The construction is compatible with étale groupoid structure, i.e. for every base point $x \in X$ there is an open neighborhood $x \in U \subset X$ and for every $g \in s^{-1}(x)$ there is an open neighborhood $g \in E_g \subset G$ such that $E_g \xrightarrow[s]{\approx} U_x$. And analogous for the target map t.

Groupoid formulation

Corollary

Given two open subsets $U_k \subset U_1 \cup U_2$ that are locally simply connected. Then the fundamental groupoid of the union computes as

$$\Pi(U_1 \cup U_2) \cong \Pi(U_1) * \Pi(U_2)/N(U_1, U_2) \tag{9}$$

where $*$ is the free product of the groupoids $\Pi(U_k)$ over the joint base $U_1 \cup U_2$ and $N(U_1, U_2) \lhd \Pi(U_1) * \Pi(U_2)$ the normal subgroupoid generated by $i_1(g^{-1})i_2(g)$ for all $g \in \Pi(U_1 \cap U_2)$.

The advantage is that we can weaken the connectedness condition for $U_1 \cap U_2$.

Higher homotopy groups

Definition

Given a path connected topological space X and a positive integer $n \in \mathbb{N}^*$. The n-th homotopy group $\pi_n(X)$ based at $x \in X$ is the set of all continuous maps $f\colon [0,1]^n \to X$ such that $f(\partial[0,1]^n) = \{x\}$ modulo homotopy with fixed boundary. The group operation is $[f] * [g] = [p]$ for every pair of sphere maps $f, g\colon [0,1]^n \rightrightarrows X$ with

$$p(t_0, \mathbf{t}') := \begin{cases} f(2t, \mathbf{t}') & \text{for } 0 \le t_0 \le \frac{1}{2}, \\ g(2t_0 - 1, \mathbf{t}') & \text{for } \frac{1}{2} \le t_0 \le 1. \end{cases} \tag{10}$$

Proposition

Given a path-connected locally simply connected space X, then $\pi_n(X)$ is a discrete abelian group for $n \ge 2$.

Idea of proof

This is analogous to the fundamental group. Note that for the second operation $[f] \cdot [g] := [p_2]$ for every $[f], [g] \in \pi_2(X)$ with

$$p_2(t_0, t_1) := \begin{cases} f(t_0, 2t_1) & \text{for } 0 \leq t_1 \leq \frac{1}{2}, \\ g(t_0, 2t_1 - 1) & \text{for } \frac{1}{2} \leq t_1 \leq 1. \end{cases}$$

The two operations are both associative and interchange as $(a \cdot b) * (c \cdot d) = (a * c) \cdot (b * d)$ for every $a, b, c, d \in \pi_2(X)$. Together with $b = (1 \cdot b)$ and $c = (c \cdot 1)$ this implies $a * d = a \cdot d$ and $b * c = c \cdot b$. Therefore there is only one group operation and it is commutative. □

Example of the higher homotopy groups

0. Replacing $\mathbb{S}^0$ by a point 0, $\pi_0(X) = [X]$, i.e. the path-connected components. This is in general not a group.
1. Note that every map $\mathbb{S}^n \to \mathbb{S}^1$ for $n > 1$ is contractible to a point. Therefore $\pi_n(\mathbb{S}^1) = 1$ for $n > 1$.
2. It turns out that the continuous maps $g\colon \mathbb{S}^2 \to \mathbb{S}^2$ are characterized by a wrapping number $[g] \in \mathbb{Z}$, i.e. $g \simeq h$ iff $[g] = [h]$. Moreover the identity $g_0 := \mathrm{id}_{\mathbb{S}^2}$ generates all other continuous maps up to homotopy. Therefore $\pi_2(\mathbb{S}^2) = \langle g_0 \rangle \cong \mathbb{Z}$.
3. The other homotopy groups are harder to compute. In particular $\pi_n(X)$ can be nonzero even if X is a quite smooth space (e.g. a topological manifold) of dimension $\dim X < n$.